1908MNRAS..68..460T

## *Note on the condition for the passage of the Earth through the plane of Saturn's Ring.* By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. The interesting observations made recently on the ring seen edgewise have brought several inquiries as to the recurrence of this beautiful phenomenon; and the following note, originally made some years ago in consequence of an inquiry from Mr. C. T. Whitmell, may be of use to others. The late Mr. R. A. Proctor gives in his book *Saturn and his System* a general account of the manner in which the Earth may pass through the plane of the ring, either once or three times, at each favourable opportunity. The present note gives the explanation in more compact form.

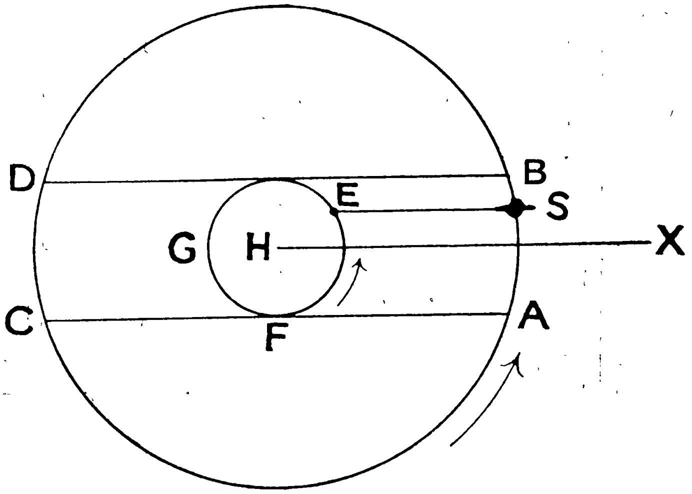

Fig. 1.

2. Let H be the Sun (fig. 1), EGF, ABDC, the orbits of the Earth and Saturn, supposed circular and in one plane. Let BD and AC be two tangents to the Earth's orbit parallel to the plane of Saturn's ring. Then if S be Saturn and E the Earth at a time of passage through the ring, ES must be parallel to BD or AC. Hence Saturn must be either in the portion AB or DC of his orbit. Since his orbit is ten times the size of the Earth's, these favourable opportunities are confined to limited periods which recur at long intervals. AB is about $\frac{1}{15}$ of the semi-orbit, and is described in about a year, so that the Earth meanwhile makes a complete revolution. If, when Saturn is near A, the Earth is near F, then there may be three passages through the ring.

3. To find the condition in exact terms, take HX parallel to AC or BD as axis of $x$. Let the radius of the Earth's orbit be unity, and that of Saturn's orbit $n^2$. The co-ordinates of the Earth may be written

$$x_1 = \cos t, \; y_1 = \sin t,$$

1908MNRAS..68..460T

and of Saturn (remembering Kepler's Third Law)

$$x_2 = n^2 \cos (t - a)/n^3, \quad y_2 = n^2 \sin (t - a)/n^3,$$

where the unit of time is $1/2\pi$ of a year and the origin of time is taken when the Earth lies on HX, Saturn then being an angle $a/n^3$ behind it.

The condition for passage of the Earth through the ring is thus

$$y_1 = y_2 \quad \text{or} \quad \sin t = n^2 \sin (t - a)/n^3,$$

a transcendental equation, which we can only solve by approximation.

4. Let us draw the curves

$$\text{(a)} \quad y = \sin t,$$

$$\text{(b)} \quad y = n^2 \sin(t - a)/n^3,$$

then the required values of $t$ will be given by their intersections. Now the curves are both sine curves, differing only in period and amplitude.

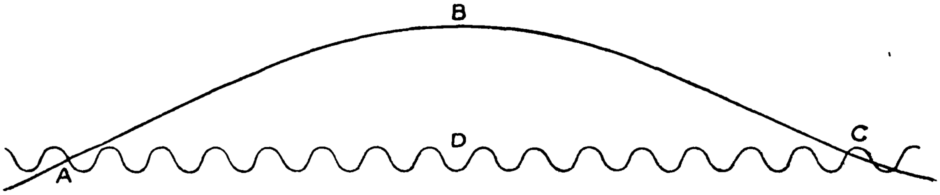

FIG. 2.

The Earth's curve (a) alternates much more rapidly, as in ADC (fig. 2). Saturn's curve (b) is a broad sweep, ABC, and we see at once how the crossings of the two curves correspond to the favourable opportunities of fig. 1; and further, that there may be a single intersection as at A, or a triple intersection as at C. There must always be an odd number of intersections; whether there can be five, seven, or more depends on the actual dimensions of the curves. If there is only one intersection, notice that the slopes of the two curves are in opposite directions, *i.e.* the planets are on opposite sides of the Sun; so that Saturn will be near conjunction with the Sun, and the phenomenon will not be so readily observable. If there are three intersections, the middle one is near opposition and the other two near quadratures.

5. In fig. 3, let ABCDE be two consecutive waves of the Earth's curve, and let FG, HK, LM represent three possible positions of a portion of Saturn's curve, each touching the Earth's curve in one point (F, K, L) and cutting it in another (G, H, M). It is clear that if Saturn's curve cuts the axis OX between T and V there will be only one intersection; if between V and W there will be three. The separating cases are thus defined by the three values of $a$, $OT = a_1$, $OV = a_2$, $OW = a_3$, and it is easily seen that $a_2 = 2\pi - a_1$, $a_3 = a_1 + 2\pi$.

1908MNRAS..68..460T

6. To find the value of $a_1$, we have the conditions that both $y$ and $\frac{dy}{dt}$ must be the same for both curves.

$$\therefore \left.\begin{aligned} \sin t &= n^2 \sin (t-a)/n^3 \\ n^3 \cos t &= n^2 \cos (t-a)/n^3 \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

$$\therefore \quad \sin^2 t + n^6 \cos^2 t = n^4 \quad . \quad . \quad . \quad . \quad (3)$$

$$n^6 \sin^2 (t-a)/n^3 + \cos^2 (t-a)/n^3 = n^2 \quad . \quad . \quad (4)$$

whence $\qquad \cos^2 t = (n^4 - 1)/(n^6 - 1)$

so that $\qquad \cos t = \frac{1}{n}\left\{1 + \frac{1}{n^4+n^2}\right\}^{-\frac{1}{2}} \quad . \quad . \quad (5)$

and further $\qquad \sin (t-a)/n^3 = \frac{1}{n^2}\left\{1 + \frac{n^2+1}{n^4}\right\}^{-\frac{1}{2}} \quad . \quad . \quad (6)$

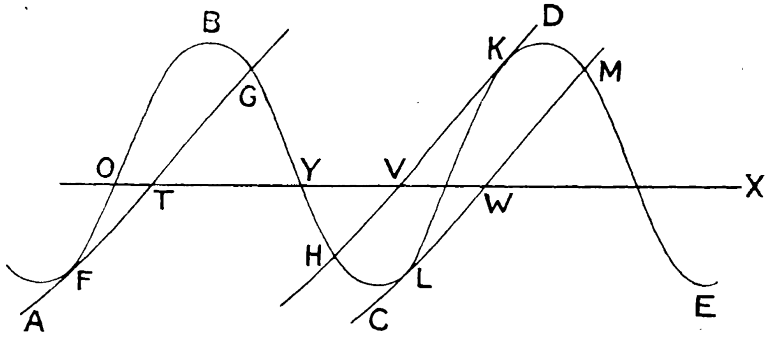

Fig. 3.

Since $n^3$ is 29·46, the number of years in which Saturn revolves round the Sun, $n = 3{\cdot}088$, $n^2 = 9{\cdot}54$, $n^4 = 91{\cdot}0$. Neglecting quantities of the order $n^{-4}$ in $t$,

$$t = -\frac{\pi}{2} + \frac{1}{n} + \frac{1}{6n^3}, \quad (t-a) = -n + \frac{1}{2n} - \frac{1}{24n^3}$$

$$\therefore \quad a = -\frac{\pi}{2} + n + \frac{1}{2n} - \frac{5}{24n^3} \quad . \quad . \quad . \quad (7)$$

Thus for Saturn-Earth $a = 1{\cdot}683$, or 96° 25′. Or we can, of course, obtain the numerical values of $t$ and $(t-a)/n^3$ directly from equations (5) and (6).

7. Hence $\qquad a_1 = 96^\circ, \quad a_2 = 264^\circ, \quad a_3 = 456^\circ.$

And in the long-run, since the period of Saturn is incommensurable with that of the Earth, the values of $a$ will be distributed uniformly, so that in a long series of years the chance of three intersections is

$$(a_3 - a_2)/360^\circ = 192/360 = {\cdot}53,$$

and the chance of one intersection is

$$(a_2 - a_1)/360^\circ = 168/360 = {\cdot}47.$$

1908MNRAS..68..460T

Starting from any crossing with a value $\alpha_0$ for $\alpha$, the next crossing *in the same direction* will be at a point

$$\frac{t-\alpha_0}{n^3}=2\pi$$

$$\text{or} \qquad t=\alpha_0+29{\cdot}46\times 2\pi$$

$$=\alpha_0+166^\circ \text{ omitting } 29\times 2\pi.$$

The intermediate crossing is in the opposite direction and at the opposite point of the orbit. The eccentricity of Saturn's orbit is so large that the intermediate value of $\alpha$ cannot be inferred by elementary considerations. There will be practically two separate series of values of $\alpha$ referring to the two nodes, each increasing by the difference 166°. But this value 166° is only approximate, and is modified by the motion of the ring plane. It appears from the subjoined table that at present the average value is about 162°.

8. The dates when Saturn crosses the line of nodes, or more properly when his ring-plane passes through the Sun, are given by Mr. Proctor on p. 223 (Table X.) of his book above mentioned (*Saturn and his System*). It will suffice here to give a few of them to the nearest day :—

| Year. | Sun in plane of Ring. | Earth's Passage. | Fractions of Year. Sat. | Earth. | Diffce. | | α. | |
|---|---|---|---|---|---|---|---|---|
| 1789 | Oct. 7 | Sept. 13 | ·764 | ·698 | ·066 | ... | 24° | ... |
| 1803 | June 23 | Mar. 12 | ·473 | ·192 | ... | ·281 | ... | 101° |
| 1819 | Mar. 23 | Sept. 14 | ·221 | ·700 | ·521 | ... | 187 | ... |
| 1832 | Dec. 4 | Mar. 12 | ·923 | ·192 | ... | ·731 | ... | 263 |
| 1848 | Sept. 3 | Sept. 14 | ·671 | ·700 | ·971 | ... | 349 | ... |
| 1862 | May 18 | Mar. 12 | ·375 | ·192 | ... | ·183 | ... | 66 |
| 1878 | Feb. 14 | Sept. 15 | ·120 | ·703 | ·417 | ... | 150 | ... |
| 1891 | Oct. 29 | Mar. 13 | ·824 | ·195 | ... | ·629 | ... | 226 |
| 1907 | July 29 | Sept. 16 | ·572 | ·706 | ·866 | ... | 312 | ... |

Columns 1 and 2 above are from Mr. Proctor's book; column 3 is filled in from the *Nautical Almanac* by finding the date when the Earth had the same heliocentric longitude as Saturn had on the date given in the column before. The next two columns give these dates in fractions of a year, and the differences are the values of $\alpha$ in the unit adopted in § 3, viz. $1/2\pi$ of a year, more conveniently expressed in degrees in the next columns. The two series are kept separate, and it will be seen that the common difference is about 162° or 163°. If it were exactly $162^\circ=180^\circ-18^\circ$, the numbers would repeat after twenty terms, and we should have the series as follows :—

1908MNRAS..68..460T

*First Series.*

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Year | 1789 | 1819 | 1848 | 1878 | 1907 | 1937 | 1966 | 1996 | 2025 | 2055 | 2084 |
| $\alpha$ | 25° | 187° | 349° | 151° | 313° | 115° | 277° | 79° | 241° | 43° | 205° etc. |
| No intersectns. | 3 | 1 | 3 | 1 | 3 | 1 | 3 | 3 | 1 | 3 | 1 |

*Second Series.*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Year | 1803 | 1832 | 1862 | 1891 | 1921 | 1950 | 1980 | 2009 |
| $\alpha$ | 101° | 263° | 65° | 227° | 29° | 191° | 83° | 245° etc. |
| No intersectns. | 1 | 3 | 3 | 1 | 3 | 1 | 3 | 1 |

Under the actual conditions, the series will slowly diverge from these, just as the cycle of total solar eclipses on the Earth slowly changes.

9. If the slope of Saturn's curve were less relatively to that of the Earth, there might be five intersections. For observers on Venus or Mercury this might be the case. The limiting case when five intersections are just possible is when a Saturn curve through the point Y (fig. 3), just touches both the adjacent curves near F and K, and then equations (2) are satisfied by $\alpha = \pi$.

$$\therefore \text{ from equation (7) } \quad n = \frac{3\pi}{2} - \frac{1}{2n} - \frac{5}{24n^3} = 4{\cdot}54 \text{ approx.}$$

For values of $n$ greater than 4·54 there will accordingly be three or five intersections, and five will occur with greater frequency as $n$ increases.

Now Bode's Law gives for the successive relative values of $n^2$

| | | | | |
|---|---|---|---|---|
| 4 | 7 | 10 | 100 | [388] |
| Mer. | Ven. | Earth. | Saturn. | Neptune. |

Thus for Saturn-Venus, $n^2 = 14{\cdot}3$, $n = 3.8$

for Saturn-Mercury, $n^2 = 25$, $n = 5{\cdot}0$

Hence Saturn's ring can disappear five times to Mercury, but not to Venus. The value of $n$ for Saturn-Neptune is too small for any but three intersections. It will readily be seen that the above procedure is applicable to the outer planets, the curves interchanging character.

10. Although Saturn's ring is a special problem, there are cases somewhat similar, *e.g.* the changes in declination of another planet. Times when the declination vanishes are times when the planet passes through the Earth's equator, and thus would be in the plane of its ring if it had one. The three intersections can readily be traced in the tabulated declinations of (say) Venus.

1908MNRAS..68..460T

## *The Variability of the Nucleus of the Planetary Nebula* N.G.C. 7662. 1860·0, $\alpha$ 23^h^ 19^m^ 11^s^, $\delta$ + 41° 46′·0.

By E. E. Barnard. (Plate 12.)

This remarkable nebula (N.G.C. 7662) is situated in Andromeda. In an ordinary telescope it looks like a small round disc of a bluish-green colour. In the 4-inch finder of the 40-inch telescope it is only distinguishable from an ordinary star of the $8^m$ or $8\frac{1}{2}^m$ by a slight fuzziness of the image. In the great telescope, however, under good conditions, it is a beautiful object —a slightly elliptical disc with quite sharply defined outlines. Unsymmetrically placed on this is a roughly elliptical broken ring of greater brightness. The interior of this ring is dark, but not black, and in this, approximately central, is ordinarily a faint stellar nucleus. The broken ring (broken on its n.p. side, and with a smaller break opposite this) is of a bright bluish-green colour, while the general disc of the nebula often has the appearance of a warm or faintly reddish tint. The northern part of this inner ring is the brightest part of the nebula. The nucleus, with which I am going to deal specially, is sometimes of a bright yellowish colour, though ordinarily it is faint and apparently colourless. There is a faint star of the 15·5 or 16 magnitude exactly on the following edge of the outer disc.

Shortly after beginning to observe this nebula in 1897 I suspected that the nucleus was variable (see *Astrophysical Journal*, xiv. p. 153). Familiarity with the changing aspect of a star involved in nebulosity, due to moonlight, bad seeing, etc., has made me extremely cautious in this matter. I have, however, for several years, established conclusively the fact that the nucleus of 7662 is actually variable to an extent of upwards of three magnitudes. At times it has appeared as a bright yellowish star of about the 12th magnitude; and at other times, equally favourable, it has been either entirely invisible or excessively faint. Though I have not definitely determined the period, the light changes seem to cover about 28 days. The light curve is still undetermined; the normal condition, however, seems to be faint—the nucleus remaining bright for a few days only.

I have collected here the principal observations of the brightness of the nucleus, etc., obtained with the 40-inch telescope in the past ten years.

On 1904 August 6, among other objects, I showed this nebula to Professor H. H. Turner. Though we both looked very carefully, under fairly good conditions, we could not see any trace of the nucleus. It must have been below the 16th magnitude at that time.

In all the observations of this nebula the eyepiece was adjusted for the nebula itself, the focus of which, as I have shown elsewhere (*M.N.*, lx. pp. 255–257, and *Astrophysical Journal*, xiv. pp. 151–157), is about a quarter of an inch outside that for a star.